Asteroid orbital ranging using Markov-Chain Monte Carlo

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Abstract–We present a novel Markov-Chain Monte-Carlo orbital ranging method (MCMC) for poorly observed single-apparition asteroids with two or more observations. We examine the Bayesian a posteriori probability density of the orbital elements using methods that map a volume of orbits in the orbital-element phase space. In particular, we use the MCMC method to sample the phase space in an unbiased way. We study the speed of convergence and also the efficiency of the new method for the initial orbit computation problem. We present the results of the MCMC ranging method applied to three objects from different dynamical groups. We conclude that the method is applicable to initial orbit computation for near-Earth, main-belt, and transneptunian objects.

INTRODUCTION

Initial asteroid orbit computation, particularly in the case of short orbital arcs, constitutes a special problem in astronomy. In recent decades, many nonlinear probabilistic methods have been developed for single-apparition asteroids, due to the importance of uncertainty estimation. Orbit computation has, however, a long history, beginning with the so-called deterministic techniques.

From the early 19th century, when the first asteroid was found, asteroid discovery methods have dramatically improved along with the development of more efficient observing instruments. With the technical improvement and development of observation techniques came a corresponding improvement in orbit computation methods (Bowell et al. 2002). An increasing number of primarily ground-based surveys have resulted in an exponential growth in the number of discoveries. The need to search for new asteroids has recently been strengthened by the awareness of the possibility of Earth impact by these objects, underscored by the recent Earth impact by 2008 TC3. As of August 6, 2009, approximately 64 370 000 observational records in total (including both comets and asteroids) are stored at the Minor Planet Center (MPC). Out of these about 63 900 000 are asteroid observational records, with about 53 400 000 for numbered and 10 500 000 for unnumbered asteroids. Every year, large numbers of asteroids are discovered, for which initial orbit computation is needed in light of the collision probability problem. Conventional, deterministic methods of preliminary orbit computation often come with vast restrictions for their usage (e.g., number of required observations), are based on many assumptions (e.g., observations made at perihelia), and can still simply fail to produce results for the object. Therefore, a need for novel, reliable, and faster methods has become apparent.

Statistical asteroid-orbit-computation methods have proved to be important in problems such as computing the collision probability (Muinonen et al. 2001), performing dynamical classification (Virtanen et al. 2008), identifying asteroids (Granvik and Muinonen 2005, 2008), and aiding the recovery of lost objects (Virtanen et al. 2002). However, these techniques can be optimized to achieve better speed of convergence to the target density and efficiency in exploring the entire density. Here we present a novel variant of these methods, evolving from the statistical ranging technique (Virtanen 2005) and making use of the Markov-Chain Monte-Carlo method (MCMC) (Robert and Casella 2004).

In MCMC Ranging section, we describe the solution of the orbital inverse problem using the Bayesian treatment, and then continue by outlining the numerical methods used for obtaining the final marginal a posteriori probability densities. In particular, we focus on the novel method. In section 3, we apply the method to altogether three objects from three different dynamical groups: near-Earth objects (NEOs), main-belt objects (MBOs), and transneptunian objects (TNOs). We compare the results to those obtained with Monte-Carlo (MC) statistical ranging (e.g., Virtanen et al. 2001). Finally, we summarize our findings in the Conclusions section.
ORBITAL INVERSE PROBLEM

Orbital-Element Probability Density

In our Bayesian treatment of orbit computation (Muinonen and Bowell 1993; Virtanen et al. 2001; Muinonen et al. 2001; Granvik and Muinonen 2005), we adopt the same notation as before, that is, we denote by:

- $\Psi = (\alpha, Q, t)$ — a set of $N$ observations, that is, pairs of Right Ascensions (R.A.) and Declinations (Dec.), obtained at the corresponding observation dates $t = (t_1; \ldots; t_N)^T$,
- $\Phi(P, t)$ — the corresponding computed sky-plane positions at the dates $t = (t_1; \ldots; t_N)^T$,
- $P = (x, y, z, \dot{x}, \dot{y}, \dot{z})^T$ — the osculating orbital elements, that is, the position and velocity, at the epoch $t_0$. For illustration purposes, we also use the Keplerian elements $(a, e, I, \omega, M_0)$, where the elements are: the semimajor axis $(a)$, eccentricity $(e)$, inclination $(i)$, longitude of ascending node $(\Omega)$, argument of perihelion $(\omega)$, and the mean anomaly $(M_0)$.

As previously, our goal is the characterization of the a posteriori probability density of the elements (p.d.f.) $p_P$:

$$p_P(P) = C p_p(P) p_e(\Delta \Psi(P)),$$

where $p_p(P)$ is the a priori p.d.f., $p_e(\Delta \Psi(P))$ is the observational error p.d.f. (usually being assumed Gaussian), evaluated for the observed-minus-computed (O-C) residuals $\Delta \Psi(P)$, including all observations. $C = \left[ \int p_p(P) dP \right]^{-1}$ is the normalization constant, and $p(P, \Psi) = p_p(P)p_e(\Delta \Psi(P))$.

For the mathematical form of $p_p$ to be invariant in transformations from one orbital element set to another, one can use, e.g., Jeffreys’ a priori p.d.f. (Jeffreys 1961):

$$p_p(P) \propto \sqrt{\det \Sigma^{-1}(P)},$$

$$\Sigma^{-1}(P) = \varphi(P)^T \Lambda^{-1} \varphi(P),$$

where $\Sigma^{-1}$ is the information matrix (or the inverse covariance matrix) evaluated for the local orbital elements $P$, $\varphi$ contains the partial derivatives of R.A. and Dec. with respect to the orbital elements, and $\Lambda$ is the covariance matrix for the observational errors.

The final a posteriori orbital-element p.d.f. (including Jeffreys’ a priori) is (Virtanen 2005):

$$p_p(P) \propto \sqrt{\det \Sigma^{-1}(P)} \exp \left( \frac{1}{2} \chi^2(P) \right),$$

$$\chi^2(P) = \Delta \Psi^T(P) \Lambda^{-1} \Delta \Psi(P).$$

For the present orbital inversion problem, we omit Jeffreys’ a priori p.d.f., because of its dependence on the dates of the observations, and use constant a priori for the Cartesian elements. In practice, the a posteriori density for the orbital elements can be very complicated. When other methods fail, a Monte-Carlo method can be used to estimate the uncertainties in the parameters of the inversion problem. In particular, here we use the MCMC method to characterize complicated a posteriori densities.

MCMC RANGING

The new method, Markov-Chain Monte-Carlo ranging (MCMC ranging) uses the Metropolis-Hastings algorithm (M-H) (Robert and Casella 2004) and is based on MC ranging. First, as in MC ranging from a given set of observations, two observations (typically the first and the last, denoted by A and B respectively) are chosen. Then, the corresponding topocentric distances (ranges $\rho_A$ and $\rho_B$) as well as the R.A. ($\alpha_A$ and $\alpha_B$) and Dec. ($\delta_A$ and $\delta_B$) angles are sampled. We start the MCMC mapping of the orbital-element phase space by introducing Gaussian proposal densities $p(Q'|Q)$ for the set of two spherical positions $Q = (\rho, \alpha, \delta)$ and $Q' = (\rho', \alpha', \delta')$ which correspond to complicated proposal densities in the orbital element phase space. We denote the candidate (proposal) and the last accepted spherical positions by $Q'$ and $Q$, respectively. We use the proposal densities to generate a chain of successive orbital elements $(P', \ldots; P^{(v)})$. In practice, we generate candidate ranges $(\rho_A', \alpha_A', \delta_A')$ and $(\rho_B', \alpha_B', \delta_B')$ for the two chosen observation dates with the help of the proposal densities. Then, as in MC ranging from the two spherical coordinates sampled at dates A and B, we calculate an orbit $P'$ and monitor the fit to all observations. The candidate orbit $P'$ is accepted or rejected based on the M-H acceptance criteria. If the orbit is accepted, we center our proposal densities on the
spherical coordinates corresponding to that orbit. This process is repeated until the stationary a posteriori density is reached.

The new candidate orbital elements \( \mathbf{P}' \) are accepted or rejected depending on the value of the p.d.f. corresponding to the candidate orbital elements \( p_p(\mathbf{P}') \), the p.d.f. corresponding to the orbital elements of last accepted sample \( p_p(\mathbf{P}_t) \), and the ratio of orbital-elements proposal p.d.f.s. In practice, we calculate a value \( a_r \):

\[
a_r = \frac{p_p(\mathbf{P}') p_p(\mathbf{Q}, \mathbf{Q}')}{p_p(\mathbf{P}) p_p(\mathbf{Q}', \mathbf{Q})} J_t',
\]

where \( J' \) and \( J_t \) are the determinants of Jacobians from topocentric coordinates to orbital parameters for the candidate

Fig. 1. A sample of 5000 different orbit solutions for TNO 2002 CX224 (observational time interval 20.98 d) obtained with MCMC ranging. The following proposal standard deviations were used: \( \sigma_{\alpha_A} = \sigma_{\delta_A} = \sigma_{\alpha_B} = \sigma_{\delta_B} = 0.5'' \) for the angular coordinates, \( \sigma_{\rho_A} = \sigma_{\rho_B} = 2.0 \) AU for the ranges, and high correlation \( \text{Cor} (\rho'_A, \rho'_B) = 0.999 \) between the ranges.

Fig. 2. Distribution of ranges for the observation dates A and B for TNO 2002 CX224 (observational time interval 20.98 d) obtained with MCMC ranging (Parameters as in Fig. 1).
and the last accepted sample, respectively. The Jacobians are defined as:

\[ J = \det \begin{bmatrix} Q \frac{\partial}{\partial P} \end{bmatrix}. \]  

(6)

Including the Jacobians in order to sample in the orbital-element phase space guarantees that the stationary density is unbiased. In MCMC ranging, we make use of a symmetric proposal p.d.f., \( p(Q'; Q) = p(Q'; Q_t) \), therefore \( a_r \) can be further reduced to:

\[ a_r = \frac{p_t(P)}{p_t(P')} J. \]  

(7)

We accept the proposed candidate elements if the value \( a_r \) is larger than or equal to 1. If that value is smaller than 1, we accept the proposed elements with a probability equal to \( a_r \):

\[ \begin{cases} P_{t+1} = P' \text{ with probability } a_r \\ P_{t+1} = P \text{ with probability } 1-a_r \end{cases} \]  

(8)

In practice, this means that a trial orbit is always accepted if it produces a better fit to the full observational data set than the last accepted one. The trial orbit is sometimes accepted if the fit is worse than the last accepted solution.

The algorithm is run for a large number of iterations until the entire possible orbital-element space is mapped. To shorten the burn-in phase, we start our mapping with a high probability orbit obtained from any orbit computation method (least squares, Gauss, statistical ranging, etc.). Similarly to MC statistical ranging, this method has a possibility to include Bayesian a priori information (such as lower and upper limits in semimajor axis, perihelion and aphelion distance, range, etc.) and enquires for elliptic orbit solutions. The minimum default geocentric distance is 1 Earth radius.

**RESULTS AND DISCUSSION**

We study the efficiency and speed of convergence of MCMC ranging for the orbit computation problem, in order to optimize existing techniques. Observational data for three objects from three different dynamical groups are being used, namely: NEO 2004 HA<sub>39</sub>, MBO 2004 QR, and TNO 2002 CX<sub>224</sub>. Observations for these objects are available from the Minor Planet Center. The full observational data sets for the objects are described in Table 1. In order to use our initial orbit computation methods (applicable to objects with short observational arcs), from the full observational data sets, we have selected parts of the observations for our computation. These data sets are also described in Table 1.

We have applied the MCMC ranging method to 2002 CX<sub>224</sub> and obtained densities of possible orbital elements, see Fig. 1 (for illustration purposes, we use the Keplerian elements). Because of the relatively short observational intervals, we have decided to use a 2-body dynamical model, which is accurate enough for this study. The inversion epoch of the orbital elements corresponds to the epoch of the first observation. For the ranges, we have used a bivariate Gaussian proposal density with standard deviation \( \sigma_a = \sigma_B = 2.0 \) AU and a high correlation between ranges \( Cor(P_r, P_g) = 0.999 \), and for the R.A. and Dec we have used Gaussian proposals with \( \sigma_a = \sigma_B = 0.5'' \). The choice of the bivariate proposal density for the ranges is desirable as it mimics the bivariate density for the ranges in the case of objects with short observational arcs (see Fig. 2) and therefore results in better efficiency of sampling.

To validate the results, we compare them with the results from MC statistical ranging for each object. There are no visible differences in the obtained marginal probability densities of the orbital elements as compared to the results of ranging. There is no significant difference between the cumulative density functions (c.d.f.s) obtained with the two methods. We also check our solutions against the least-squares solution derived from the full set of observations (see Table 2) in the n-body approach. The orbital-element densities presented here cover the least-squares orbit solutions derived from the full observation sets.

Although the MCMC ranging method has proven in practice to be an efficient and fast method of initial orbit computation, it depends on the tuning of the proposal densities. The choice of the proposal densities plays a crucial role in obtaining a good coverage during the exploration of the phase space. We highlight this problem, by illustrating the coverage of the phase space, for different proposal standard deviations for the ranges in the case of 2002 CX<sub>224</sub> in Fig. 3. For all of these cases, the standard deviation of the proposal density for the angular coordinates was chosen to be \( \sigma_\alpha = \sigma_B = \sigma_\delta = \sigma_B = 0.5'' \). This deviation depends on the observational noise, which is here assumed to be 0.5'' for both R.A.s and Dec.s. If the proposal deviation in the sky-plane coordinates is too small, we start to observe chain patterns appearing in the distribution of the O-C residuals (see Fig. 4 for an example). If the proposal standard deviation of ranges is too small, we obtain high orbit acceptance ratio but poor coverage of the orbital-element phase space. If this value is too large, then the acceptance ratio is not satisfactory. Somewhere in between lies the proper choice of the proposal

**Table 2. Least-squares orbits.**

<table>
<thead>
<tr>
<th>Object</th>
<th>2004 HA&lt;sub&gt;39&lt;/sub&gt;</th>
<th>2004 QR</th>
<th>2002 CX&lt;sub&gt;224&lt;/sub&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>a (AU)</td>
<td>2.1493040(34)</td>
<td>2.33155(74)</td>
<td>46.404(57)</td>
</tr>
<tr>
<td>e</td>
<td>0.5368719(72)</td>
<td>0.30163(21)</td>
<td>0.1329(49)</td>
</tr>
<tr>
<td>i (°)</td>
<td>36.2085(3)</td>
<td>6.3566(36)</td>
<td>16.8447(63)</td>
</tr>
<tr>
<td>Ω (°)</td>
<td>204.154337(44)</td>
<td>314.82(1)</td>
<td>42.2592(56)</td>
</tr>
<tr>
<td>ω (°)</td>
<td>67.3216(11)</td>
<td>330.701(26)</td>
<td>132.51(1.07)</td>
</tr>
<tr>
<td>M&lt;sub&gt;0&lt;/sub&gt; (°)</td>
<td>345.02807(39)</td>
<td>28.4223(89)</td>
<td>256.12(1.73)</td>
</tr>
</tbody>
</table>

The 1σ standard deviations for the orbital elements are given after each element in the units of the last digit shown.
Fig. 3. Three sets of distributions of orbital elements (each containing 5000 different orbit solutions) for TNO 2002 CX224 (observational time interval 20.98 d) obtained with MCMC ranging with varying proposal standard deviation of the ranges: a) $\sigma_\rho^A = \sigma_\rho^B = 0.01$ AU, b) $\sigma_\rho^A = \sigma_\rho^B = 0.1$ AU, c) $\sigma_\rho^A = \sigma_\rho^B = 1.0$ AU, d) $\sigma_\rho^A = \sigma_\rho^B = 10.0$ AU. Proposal standard deviation for angular coordinates was $\sigma_\alpha^A = \sigma_\delta^A = \sigma_\alpha^B = \sigma_\delta^B = 0.5'$ and the correlation between the ranges is $\text{Cor}(\rho, \rho) = 0.999$ in all the cases. Poor phase-space coverage in (a) and (b), good coverage in (c) and (d). We denote the argument of perihelion by using the $\omega$ symbol and the longitude of the ascending node by using the $\Omega$ symbol.
standard deviation of the ranges, which results in good acceptance and coverage. This choice can be, however, difficult to make, e.g., in cases when the resulting coverage and acceptance ratios are similar. In the example of the TNO, we have found that \(0.5 < (\sigma_{\rho_A} = \sigma_{\rho_B} < 8.0\) AU give both good coverage and acceptance. The results seems to be less sensitive to changes in the proposal standard deviation of the angular coordinates than changes in the standard deviation of ranges. In general, \(\sigma_{\rho_A}\) and \(\sigma_{\rho_B}\) have to be tuned individually for each object, but some dependence of that choice on the length of the observational arc and dynamical group affiliation have been noticed.

First guesses for the ranges of the object (if available) at the observation dates A and B could help to establish the size of the proposal density for the ranges. The present method might also have difficulties in solving the special cases of multi-modal probability densities. However, this could be partially solved, e.g., by the usage of more than one chain (e.g., some tens of separate chains), each starting with different orbit in the first step. All chains should be subjected to convergence diagnostics.

The present method allows for direct MCMC sampling of the orbital-element phase space. We present the CPU computation time and acceptance rate with different methods for the three different objects in Table 3. For the purpose of fair comparison, in the standard ranging, we have excluded Jeffreys’ a priori. The computation times as well as the acceptance ratios are in all cases better or comparable with the ones obtained with standard statistical ranging (see Table 3). The main advantage of the method is however the possibility to perform an unbiased sampling of six-dimensional volumes with higher p.d.f. values for the sample orbits.

MCMC ranging is also applicable to objects from different dynamical groups. Here, we have also applied the new method to the NEO 2004 HA\(_{39}\) and MBO 2004 QR. We present the distribution of the orbital elements obtained for these objects in Figs. 5 and 6. The new method is now publicly available in an open-source asteroid-orbit-computation software package OpenOrb. For more details, see Granvik et al. (2009).

**CONCLUSIONS**

We have developed a novel variant of asteroid orbit computation methods that maps the extensive volume of solution space and applied it to different classes of objects. Approach presented here can successfully sample the solution space. The method performs an unbiased sampling, which contrary to standard statistical ranging is guided by the MCMC algorithm in the six-dimensional orbital-element space.
Fig. 5. A sample of 5000 different orbit solutions for MBO 2004 QR (observational time interval 2.04 d) obtained with MCMC ranging. The following proposal standard deviations were used: $\sigma_{\alpha_A} = \sigma_{\delta_A} = \sigma_{\alpha_B} = \sigma_{\delta_B} = 0.5''$ for the angular coordinates, $\sigma_{\rho_A} = \sigma_{\rho_B} = 0.3$ AU for the ranges.

Fig. 6. A sample of 5000 different orbit solutions for NEO 2004 HA39 (observational time interval 4.75 d) obtained with MCMC ranging. The following proposal standard deviations were used: $\sigma_{\alpha_A} = \sigma_{\delta_A} = \sigma_{\alpha_B} = \sigma_{\delta_B} = 0.5''$ for the angular coordinates, $\sigma_{\rho_A} = \sigma_{\rho_B} = 0.01$ AU for the ranges.
space. It results in an increased number of solutions with high p.d.f. values, while covering the same phase space. The present method has also been shown to increase the speed of convergence and the acceptance ratio.

Usage of the MCMC approach resulted in a new way of sampling of complicated densities in orbit computation problems, however the performance for individual objects depends on the tuning of the proposal densities. Our MCMC method is recommended for initial orbit computation for objects with small numbers of observations and short observational arcs. In such cases, the orbital parameter uncertainties are known to be significant, the a posteriori p.d.f.s can turn out to be complicated, and the inverse problem may be ill-posed or underdetermined.

The distribution of O-C residuals could be used in outlier detection. Including the outlier in the inversion results in a visible shift of residual cloud(s). If there is no outlier in the data, the residual clouds are aligned. The method developed here might also be useful in other problems like computing collision probabilities, dynamical classification, and performing the recovery of lost objects. We will continue to investigate inversion methods for the problematic multimodal probability densities and search for different approaches to solve this problem.

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